# PLC Chapter 04
## first-order functional language, type checking

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Note by Baris Aktemur:
These slides have been shortened and rearranged from the originals available at http://www.itu.dk/courses/BPRD/E2013/.
I thank Peter Sestoft for making the PPT's available.

# Micro-ML: A small functional language

- First-order: A value cannot be a function
- Dynamically typed, so this is OK:

  **`if true then 1+2 else 1+false`**
- Eager, or call-by-value: In a call **`f(e)`** the argument **`e`** is evaluated before **`f`** is called
- Example Micro-ML programs (an F# subset):

```
5+7

let f x = x + 7 in f 2 end

let fac x = if x=0 then 1 else x * fac(x - 1)
in fac 10 end
```

# Abstract syntax of Micro-ML

```
type expr =
   | CstI of int
   | CstB of bool
   | Var of string
   | Let of string * expr * expr
   | Prim of string * expr * expr
   | If of expr * expr * expr
   | Letfun of string * string * expr * expr
   | Call of expr * expr
```

```
let f x = x + 7 in f 2 end
```

(f, x, fBody, letBody)

```
Letfun ("f", "x", Prim ("+",Var "x",CstI 7),
         Call (Var "f",CstI 2))
```

# Lexer and parser for Micro-ML

- Lexer:
  - Nested comments, as in F#, Standard ML

```
1 + (* 33 (* was 44 *) *) 22
```

- Parser:
  - To parse applications  e1 e2 e3  correctly, distinguish atomic expressions from others

- Problem: f(x-1) parses as f(x(-1))
- Solution:
  - FunLex.fsl: make **`CSTINT`** just **`[0-9]+`** without sign
  - FunPar.fsy: add rule **`Expr := MINUS Expr`**

# Runtime values, function closures

- Run-time values: integers and functions

```
type value =
   | Int of int
   | Closure of string * string * expr * value env
```

- *Closure*: a package of a function's body and its declaration environment
- A name should refer to a *statically* enclosing binding:

```
let y = 11
in let f x = x + y
   in let y = 22 in f 3 end
   end
end
```

Should always have value 11

Evaluate as 3 + y

(f, x, x+y, [(y,11)])

# Interpretation of Micro-ML

- Constants, variables, primitives, let, if: as for expressions
- Letfun: Create function closure and bind f to it
- Function call f(e):
  - Look up f, it must be a closure
  - Evaluate e
  - Create environment and evaluate the function's body

```
let rec eval (e : expr) (env : value env) : int =
    match e with
    | ...
    | Letfun(f, x, fBody, letBody) ->
      let bodyEnv = (f, Closure(f, x, fBody, env)) :: env
      in eval letBody bodyEnv
    | Call(Var f, eArg) ->
      let fClosure = lookup env f
      in match fClosure with
         | Closure (f, x, fBody, fDeclEnv) ->
           let xVal = Int(eval eArg env)
           let fBodyEnv = (x, xVal) :: (f, fClosure) :: fDeclEnv
           in eval fBody fBodyEnv
         | _ -> failwith "eval Call: not a function"
```

Evaluate fBody in declaration environment

# Dynamic scope (instead of static)

- With static scope, a variable refers to the lexically, or statically, most recent binding
- With **dynamic scope**, a variable refers to the dynamically most recent binding:

```
let y = 11
in let f x = x + y
   in let y = 22 in f 3 end
   end
end
```

Evaluate as
`3 + y`

# A dynamic scope variant of Micro-ML

- Very minimal change in interpreter:

```
let rec eval (e : expr) (env : value env) : int =
    ...
    | Call(Var f, eArg) ->
      let fClosure = lookup env f
      in match fClosure with
         | Closure (f, x, fBody, fDeclEnv) ->
           let xVal = Int(eval eArg env)
           let fBodyEnv = (x, xVal) :: (f, fClosure) :: env
           in eval fBody fBodyEnv
```

Evaluate fBody in call environment

- fDeclEnv is ignored; function is just (f, x, fBody)
- Good and bad:
  – simple to implement (no closures needed)
  – makes type checking difficult
  – makes efficient implementation difficult
- Used in macro languages, and Lisp, Perl, Clojure

# Evaluation by logical rules

$$\frac{}{\rho \vdash i \Rightarrow i}\ (e1)$$

$$\frac{}{\rho \vdash b \Rightarrow b}\ (e2)$$

$$\frac{\rho(x) = v}{\rho \vdash x \Rightarrow v}\ (e3)$$

In environment ρ, expression x evaluates to *v*

$$\frac{\rho \vdash e_1 \Rightarrow v_1 \qquad \rho \vdash e_2 \Rightarrow v_2 \qquad v = v_1 + v_2}{\rho \vdash e_1 + e_2 \Rightarrow v}\ (e4)$$

$$\frac{\rho \vdash e_1 \Rightarrow v_1 \qquad \rho \vdash e_2 \Rightarrow v_2 \qquad b = (v_1 < v_2)}{\rho \vdash e_1 < e_2 \Rightarrow b}\ (e5)$$

$$\frac{\rho \vdash e_r \Rightarrow v_r \qquad \rho[x \mapsto v_r] \vdash e_b \Rightarrow v}{\rho \vdash \mathbf{let}\ x = e_r\ \mathbf{in}\ e_b\ \mathbf{end}\ \Rightarrow v}\ (e6)$$

$$\frac{\rho \vdash e_1 \Rightarrow \mathbf{true} \qquad \rho \vdash e_2 \Rightarrow v}{\rho \vdash \mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3 \Rightarrow v}\ (e7t)$$

$$\frac{\rho \vdash e_1 \Rightarrow \mathbf{false} \qquad \rho \vdash e_3 \Rightarrow v}{\rho \vdash \mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3 \Rightarrow v}\ (e7f)$$

# How to read a rule

Environment (rho)

Judgment: In environment ρ, expression $e_2$ reduces to value $v_2$

Premises

$$\frac{\rho \vdash e_1 \Rightarrow v_1 \qquad \rho \vdash e_2 \Rightarrow v_2 \qquad b = (v_1 < v_2)}{\rho \vdash e_1 < e_2 \Rightarrow b}$$

Conclusion

- IF
  - in environment ρ, expression $e_1$ reduces to $v_1$, and
  - in environment ρ, expression $e_2$ reduces to $v_2$, and
  - b is whether $v_1$ is less then $v_2$
- THEN
  - in environment ρ, expression $e_1$<$e_2$ reduces to b

# Joint exercise: How read these?

$$\frac{}{\rho \vdash i \Rightarrow i}$$

$$\frac{\rho \vdash e_1 \Rightarrow v_1 \qquad \rho \vdash e_2 \Rightarrow v_2 \qquad v = v_1 + v_2}{\rho \vdash e_1 + e_2 \Rightarrow v} \, (e4)$$

$$\frac{}{\rho \vdash b \Rightarrow b}$$

$$\frac{\rho \vdash e_1 \Rightarrow v_1 \qquad \rho \vdash e_2 \Rightarrow v_2 \qquad b = (v_1 < v_2)}{\rho \vdash e_1 < e_2 \Rightarrow b} \, (e5)$$

$$\frac{\rho(x) = v}{\rho \vdash x \Rightarrow v}$$

$$\frac{\rho \vdash e_r \Rightarrow v_r \qquad \rho[x \mapsto v_r] \vdash e_b \Rightarrow v}{\rho \vdash \mathtt{let}\ x = e_r\ \mathtt{in}\ e_b\ \mathtt{end} \Rightarrow v} \, (e6)$$

$$\frac{\rho \vdash e_1 \Rightarrow \mathtt{true} \qquad \rho \vdash e_2 \Rightarrow v}{\rho \vdash \mathtt{if}\ e_1\ \mathtt{then}\ e_2\ \mathtt{else}\ e_3 \Rightarrow v} \, (e7t)$$

$$\frac{\rho \vdash e_1 \Rightarrow \mathtt{false} \qquad \rho \vdash e_3 \Rightarrow v}{\rho \vdash \mathtt{if}\ e_1\ \mathtt{then}\ e_2\ \mathtt{else}\ e_3 \Rightarrow v} \, (e7f)$$

# Evaluation by logical rules: Function declaration and call

- Compare these with the **`eval`** interpreter:

$$\frac{\rho[f \mapsto (f, x, e_r, \rho)] \vdash e_b \Rightarrow v}{\rho \vdash \textbf{let } f\ (x) = e_r \textbf{ in } e_b \textbf{ end } \Rightarrow v}\ (e8)$$

$$\frac{\rho(f) = (f, x, e_r, \rho_{fdecl}) \qquad \rho \vdash e \Rightarrow v_x \qquad \rho_{fdecl}[x \mapsto v_x, f \mapsto (f, x, e_r, \rho_{fdecl})] \vdash e_r \Rightarrow v}{\rho \vdash f\ e \Rightarrow v}\ (e9)$$

- Also, note recursive evaluation of f's body

# Combining evaluation rules to trees

- Stacking logical rules on top of each other
- One rule's conclusion is another's premise
- Evaluating **`let x=1 in x<2 end => true`** in some environment ρ:

$$
\dfrac{\dfrac{}{\rho \vdash 1 \Rightarrow 1}\,(e1) \quad \dfrac{\dfrac{\rho[\mathrm{x} \mapsto 1](\mathrm{x}) = 1}{\rho[\mathrm{x} \mapsto 1] \vdash \mathrm{x} \Rightarrow 1}\,(e3) \quad \dfrac{}{\rho[\mathrm{x} \mapsto 1] \vdash 2 \Rightarrow 2}\,(e1)}{\rho[\mathrm{x} \mapsto 1] \vdash \mathrm{x} < 2 \Rightarrow \texttt{true}}\,(e5)}{\rho \vdash \texttt{let}\ \mathrm{x} = 1\ \texttt{in}\ \mathrm{x} < 2\ \texttt{end} \Rightarrow \texttt{true}}\,(e6)
$$

- The **`eval`** function implements the rules, from conclusion to premise!

# Combining evaluation rules to trees

$$
\dfrac{
  \dfrac{\dfrac{}{\rho \vdash 1 \Rightarrow 1} \qquad \dfrac{}{\rho \vdash 2 \Rightarrow 2}}{\rho \vdash 1 < 2 \Rightarrow \texttt{true}}\,(e5)
  \qquad
  \dfrac{\dfrac{\rho'(\texttt{z}) = \texttt{true}}{\rho' \vdash \texttt{z} \Rightarrow \texttt{true}}\,(t3) \qquad \dfrac{}{\rho' \vdash 3 \Rightarrow 3}}{\rho[\texttt{z} \mapsto \texttt{true}] \vdash \texttt{if z then 3 else 4 end} \Rightarrow 3}\,(e7t)
}{\rho \vdash \texttt{let z = (1 < 2) in if z then 3 else 4 end} \Rightarrow 3}\,(e6)
$$

**Fig. 4.5** Evaluation of `let z = (1 < 2) in if z then 3 else 4 end` to 4. For brevity we write $\rho'$ for the environment $\rho[\texttt{z} \mapsto \texttt{true}]$

# Type Checking

- Type checking: making sure that operators, functions, names, etc. are used properly.E.g:
  - Addition is done between integers
  - Functions are applied on correct number of arguments of correct types
  - No use of undefined names
- Catch errors early, before runtime.

```
if (random() = 823857)
then junk(5(42))
else max(5, 42)
```

- Typed Micro-ML.
  - No inference (yet). Annotate functions with types.

# An explicitly typed fun. language

```
type typ =
   | TypI
   | TypB
   | TypF of typ * typ
```

```
let f (x : int) : int = x+1
in f 12 end
```

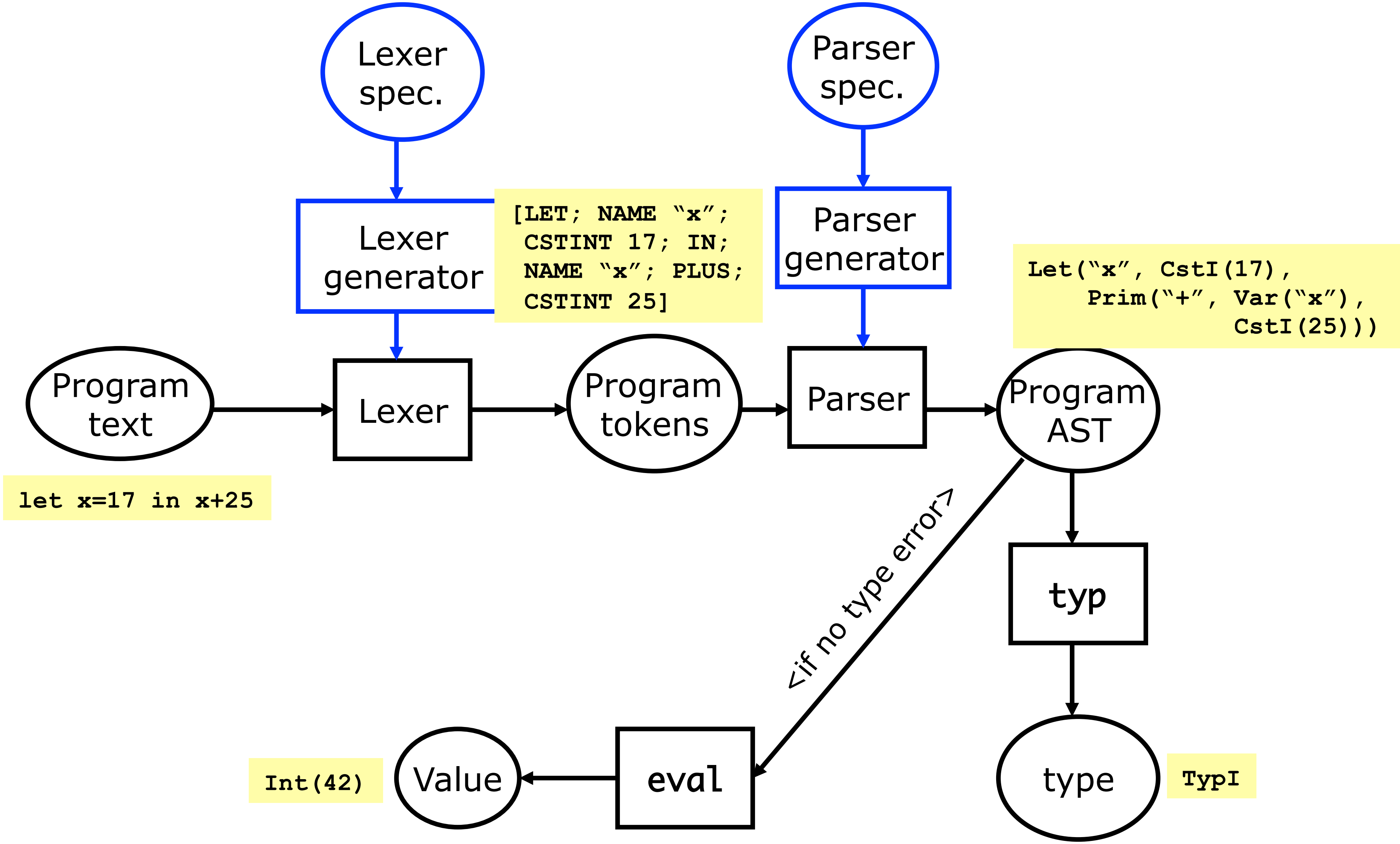
```
Letfun("f", "x", TypI,
       Prim("+", Var "x", CstI 1), TypI,
       Call(Var "f", CstI 12));;
```

(TypF(TypI, TypI)

```
type tyexpr =
   | CstI of int
   | CstB of bool
   | Var of string
   | Let of string * tyexpr * tyexpr
   | Prim of string * tyexpr * tyexpr
   | If of tyexpr * tyexpr * tyexpr
   | Letfun of string * string * typ * tyexpr * typ * tyexpr
   | Call of tyexpr * tyexpr
```

(f, x, xTyp, fBody, rTyp, letBody

# When does type checking happen?



# Type checking versus evaluation

- The type checker **`typ`** and the interpreter **`eval`** have similar structure
- Type checking can be thought of as *abstract interpretation* of the program
- We calculate "TypI + TypI gives TypI" instead of "Int 3 + Int 5 gives Int 8"
- One major difference:
  - Type checking a function call f(e) does not require type checking the function's body again
  - Interpreting a function call f(e) does require interpreting the function's body
- Type checking always terminates

# Type checking by logical rules

$$\rho \vdash i : \texttt{int}$$

$$\rho \vdash b : \texttt{bool}$$

$$\frac{\rho(x) = t}{\rho \vdash x : t}$$

$$\frac{\rho \vdash e_1 : \texttt{int} \qquad \rho \vdash e_2 : \texttt{int}}{\rho \vdash e_1 + e_2 : \texttt{int}}$$

$$\frac{\rho \vdash e_1 : \texttt{int} \qquad \rho \vdash e_2 : \texttt{int}}{\rho \vdash e_1 < e_2 : \texttt{bool}}$$

$$\frac{\rho \vdash e_r : t_r \qquad \rho[x \mapsto t_r] \vdash e_b : t}{\rho \vdash \texttt{let}\ x = e_r\ \texttt{in}\ e_b\ \texttt{end}\ : t}$$

$$\frac{\rho \vdash e_1 : \texttt{bool} \qquad \rho \vdash e_2 : t \qquad \rho \vdash e_3 : t}{\rho \vdash \texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ e_3 : t}$$

$$\frac{\rho[x \mapsto t_x, f \mapsto t_x \to t_r] \vdash e_r : t_r \qquad \rho[f \mapsto t_x \to t_r] \vdash e_b : t}{\rho \vdash \texttt{let}\ f\ (x : t_x) = e_r : t_r\ \texttt{in}\ e_b : t}$$

$$\frac{\rho(f) = t_x \to t_r \qquad \rho \vdash e : t_x}{\rho \vdash f\ e : t_r}$$

# How to read a type rule

Environment (rho)

Judgement: In environment ρ, expression $e_2$ has type int

**Note**: rho is a type environment, not a value environment.

$$\frac{\rho \vdash e_1 : \texttt{int} \qquad \rho \vdash e_2 : \texttt{int}}{\rho \vdash e_1 < e_2 : \texttt{bool}}$$

Premises

Conclusion

- IF
  - in environment ρ, expression $e_1$ has type int, and
  - in environment ρ, expression $e_2$ has type int
- THEN
  - in environment ρ, expression $e_1 < e_2$ has type bool

# Joint exercise: How read these?

$$\rho \vdash i : \mathtt{int}$$

An integer constant has type int

$$\frac{\rho(x) = t}{\rho \vdash x : t}$$

$$\frac{\rho \vdash e_1 : \mathtt{bool} \qquad \rho \vdash e_2 : t \qquad \rho \vdash e_3 : t}{\rho \vdash \mathtt{if}\ e_1\ \mathtt{then}\ e_2\ \mathtt{else}\ e_3 : t}$$

$$\frac{\rho \vdash e_r : t_r \qquad \rho[x \mapsto t_r] \vdash e_b : t}{\rho \vdash \mathtt{let}\ x = e_r\ \mathtt{in}\ e_b\ \mathtt{end}\ : t}$$

# Joint exercise: How read these?

$$\frac{\rho[x \mapsto t_x, f \mapsto t_x \to t_r] \vdash e_r : t_r \qquad \rho[f \mapsto t_x \to t_r] \vdash e_b : t}{\rho \vdash \texttt{let}\ f\ (x : t_x) = e_r : t_r\ \texttt{in}\ e_b : t}$$

$$\frac{\rho(f) = t_x \to t_r \qquad \rho \vdash e : t_x}{\rho \vdash f\ e : t_r}$$

# Type checking by recursive function

- Using a type environment [(“x”, TypI)]:

```
let rec typ (e : tyexpr) (env : typ env) : typ =
    match e with
    | CstI i -> TypI
    | CstB b -> TypB
    | Var x  -> lookup env x
    | Prim(ope, e1, e2) ->
      let t1 = typ e1 env
      let t2 = typ e2 env
      in match (ope, t1, t2) with
         | ("*",  TypI, TypI) -> TypI
         | ("+",  TypI, TypI) -> TypI
         | ("-",  TypI, TypI) -> TypI
         | ("=",  TypI, TypI) -> TypB
         | ("<",  TypI, TypI) -> TypB
         | ("&&", TypB, TypB) -> TypB
         | _    -> failwith "unknown primitive, or type error"
    | ...
```

# Type checking, part 2

- Checking `let x=eRhs in letBody end`
- Checking `if e1 then e2 else e3`

```
let rec typ (e : tyexpr) (env : typ env) : typ =
    match e with
    | Let(x, eRhs, letBody) ->
      let xTyp = typ eRhs env
      let letBodyEnv = (x, xTyp) :: env
      in typ letBody letBodyEnv
    | If(e1, e2, e3) ->
      match typ e1 env with
       | TypB -> let t2 = typ e2 env
                 let t3 = typ e3 env
                 in if t2 = t3 then t2
                    else failwith "If: branch types differ"
       | _    -> failwith "If: condition not boolean"
    | ...
```

# Type checking, part 3

- Checking **`let f x=eBody in letBody end`**
- Checking **`f eArg`**

```
let rec typ (e : tyexpr) (env : typ env) : typ =
    match e with
    | ...
    | Letfun(f, x, xTyp, fBody, rTyp, letBody) ->
      let fTyp = TypF(xTyp, rTyp)
      let fBodyEnv = (x, xTyp) :: (f, fTyp) :: env
      let letBodyEnv = (f, fTyp) :: env
      if typ fBody fBodyEnv = rTyp then typ letBody letBodyEnv
      else failwith "Letfun: wrong return type in function"
    | Call(Var f, eArg) ->
      match lookup env f with
        | TypF(xTyp, rTyp) ->
          if typ eArg env = xTyp then rTyp
          else failwith "Call: wrong argument type"
        | _ -> failwith "Call: unknown function"
    | Call(_, eArg) -> failwith "Call: illegal function in call"
```

# Combining type rules to trees

- Stacking type rules on top of each other
- One rule's conclusion is another's premise
- Checking **`let x=1 in x<2 end : bool`** in some environment ρ:

$$
\dfrac{\rho \vdash 1 : \mathtt{int} \qquad \dfrac{\rho[\mathrm{x} \mapsto \mathtt{int}] \vdash \mathrm{x} : \mathtt{int} \qquad \rho[\mathrm{x} \mapsto \mathtt{int}] \vdash 2 : \mathtt{int}}{\rho[\mathrm{x} \mapsto \mathtt{int}] \vdash \mathrm{x} < 2 : \mathtt{bool}}}{\rho \vdash \mathtt{let}\ \mathrm{x} = 1\ \mathtt{in}\ \mathrm{x} < 2\ \mathtt{end} : \mathtt{bool}}
$$

- The **`typ`** function implements the rules, from conclusion to premise!

# Joint exercises: Invent type rules and evaluation rules

- For $\texttt{e}_1$ `&&` $\texttt{e}_2$ (logical and)
- For $\texttt{e}_1$ `::` $\texttt{e}_2$ (list cons operator)
- For `match e with [] ->` $\texttt{e}_1$ `| x::xr ->` $\texttt{e}_2$

# Dynamically or statically typed

- Dynamically typed:
  - Types are checked during evaluation (micro-ML, Postscript, JavaScript, Python, Ruby, Scheme, …)

```
true { 11 } { 22 false add } ifelse =
```

OK, gives 11

- Statically typed:
  - Types are checked before evaluation (our typed fun. language, F#, most of Java and C#)

```
if true then 11 else 22+false
```

Compile-time type error

```
true ? 11 : (22 + false)
```

Compile-time type error

# Dynamic typing in Java/C# arrays

- For a Java/C# array whose element type is a reference type, all assignments are type-checked at runtime

```
void M(Object[] arr, Object x) {
  arr[0] = x;
}
```

Type check needed at run-time

- Why is that necessary?

```
String[] ss = new String[1];
M(ss, new Object());
String s0 = ss[0];
```